LIOUVILLE TYPE THEOREMS FOR CONFORMAL GAUSSIAN CURVATURE EQUATION

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ABSTRACT. In this note, we study Liouville type theorem for conformal Gaussian curvature equation (also called the mean field equation)

$$-\Delta u = K(x)e^u, \;\; in\ R^2$$

where $K(x)$ is a smooth function on R^2. When $K(x) = K(x_1)$ is a sign-changing smooth function in the real line R, we have a non-existence result for the finite total curvature solutions. When K is monotone non-decreasing along every ray starting at origin, we can prove a non-existence result too. We use the methods of moving planes and moving spheres.

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1. INTRODUCTION

In this paper, we study the Liouville type results for conformal Gaussian curvature equation (also called mean field equation or Lame-Emden equation)

$$-\Delta u = K(x)e^u, \;\; x = (x_1, x_2) \in R^2 \tag{1}$$

where is a smooth function on R^2. The geometrical meaning for the equation (1) is that the conformal metric $e^u dx^2$ has its scalar curvature K. We firstly consider the case when $K(x) = K(x_1)$ is a sign-changing smooth function in the real line R. This kind of problem arises from the apriori bound for solutions via the blowing up argument. In recent studies for the prescribed Gaussian curvature problem in R^2 or the mean field equations, Radial symmetry, Liouville theorems, and classification results for solutions with *finite energy* to equation (1) are obtained with other assumptions on positivity or negativity of K, see [13], [8],[2],[6], [10], and [1] therein for more references. In particular in [13], the best existence result of solutions for a class of positive functions K has been obtained. In the work [8], best existence result for a class of negative functions K has been obtained. In the work [6], the behavior at infinity of solutions with finite energy has

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been found when the function K being positive or negative has a controlled polynomial growth. When K is non-positive, the quantity

$$\alpha_1 = \sup\{\alpha \in R; \int_{R^2} |K(x)|(1+|x|^2)^\alpha dx < \infty\}$$

plays an important role. When K is positive with polynomial growth, the total curvature

$$\int_{R^2} K(x)e^{u(x)}dx$$

plays the key role. One may see [7] for more results. However, there are relative few result for the case when the function K is changing sign or with no control of the growth. We then show that Liouville type result is also true for a class of positive radially monotone functions K. Since we have two kinds of results with different assumptions of K, we state them separately.

Results One:

We assume that $K(x) = x_1$ is non-trivial and u is a smooth solution to (1) such that

$$u(x) \le u(0) = 0, \ in \ \ R^2, \ \ \int_{R^2} |K(x)|e^u dx < +\infty.$$

We remark that a smooth solution u with $|K|e^u \in L^1(R^2)$ is called *finite energy* solution. We point out that one may replace $K(x) = x_1$ by a nontrivial function with

$$\partial_{x_1} K(x_1) \ge 0, \ in \ \ R^2$$

in the result below.

One of the main purpose of this paper is to prove the following Liouville type theorem.

Theorem 1. *Under the assumptions above and the finite total integral* $A := \int_{R^2} x_1 u^u > 0$*, there is no smooth solution to (1).*

Intuitionally, one may believe this true. The reason is that using the Pohozaev identity (see (10) in appendix) with the vanishing boundary terms and $j = 1$, we can immediately get

$$0 < \int_{R^2} e^u = 0,$$

which is absurd. Hence, Theorem 1 is true by assuming suitable decay conditions for boundary terms on large balls. Instead of investigating this method, we shall use another method to prove this result. Assume u is a solution to (1). We shall first derive an apriori estimate for solutions u to (1) with finite total integral assumption. Then we use the moving plane method to show that u is monotone non-decreasing in x_1, which leads a contradiction.

As an application of the Liouville theorem above, we may guess the following

Theorem 2. *Assume $\tilde{K}$ is a sign-changing smooth function on the two sphere S^2 such that for small positive function $C_2 > 0$, there is a positive constant $C_1 > 0$ satisfying $|\nabla \tilde{K}(y)| \geq C_1 > 0$ for*

$$y \in \Gamma := \{y \in S^2; |\tilde{K}(y)| \leq C_2\}.$$

Then there is uniform constant $C > 0$ such that

$$|w(y)| \leq C, \quad y \in \Gamma$$

for all solutions w to the conformal Gaussian curvature equation

$$-\Delta_{S^2} w + 2 = \tilde{K}(y)e^w, \quad y = (y_1, y_2, y_3) \in S^2$$

This result has been proved by Chen and Li in [5]. So we shall not prove it, but we outline our formal argument of it. We remark that the simpler a priori estimate for solutions on the negative part of the function K is given in [12]. So we consider only the result near zero set of the function K. Choose $p \in S^2$ such that $K(p) < 0$ and $K(-p) > 0$. Let $x_1 = y_1/(1-y_3), x_2 = y_2/(1-y_3)$ be the inverse stereographic projection mapping $-p$ into origin in R^2. Then the spherical metric can be written as

$$\frac{4}{(1+|x|^2)^2}dx^2.$$

Let $u(x) = w(y(x)) + 2\log\frac{2}{1+|x|^2}$ and let $K(x) = \tilde{K}(y(x))$. Then we have

$$-\Delta u = K(x)e^u, \quad in\ R^2.$$

Note that

$$\int_{S^2} \tilde{K}(y)e^{w(y)}d\sigma_y = \int_{R^2} \tilde{K}(y(x))e^{w(y(x))}\frac{4dx}{(1+|x|^2)^2} = \int_{R^2} K(x)e^{u(x)}dx$$

and

$$\int_{S^2} |\tilde{K}(y)|e^{w(y)}d\sigma_y = \int_{R^2} |K(x)|e^{u(x)}dx.$$

Hence, the conformal Gaussian curvature equation is reduced into (1). Now, for any finite energy solution sequence (u_j) to (1) with $\gamma_j = \{x; u_j(x) = 0\}$, $u_j(x_j) \to +\infty$ and $dist(x_j, \gamma_j) \to 0$, let $d_j(x) = dist(x, \gamma_j)$.

One can show that for some constant α, $u_j(x) - \alpha \log d_j(x)$ is bounded in Γ_j (see also [5]). Let

$$v_j(x) = u_j(x + d_j(x)y) - \alpha \log d_j(x).$$

Then we have a sub-convergence sequence, still denoted by v_j, such that $v_j \to V$ in $C^2_{loc}(R^2)$ where V satisfying

$$-\Delta V = x_1 e^V, \quad x = (x_1, x_2) \in R^2.$$

Using Theorem 1 we find a contradiction. Hence, we have proved Theorem 2.

We remark that similar result for scalar curvature problem has been found by Chen and Li, C.Chen and C.S.Lin (see [4] for more references). One may weaken the assumption on K by allowing K to have large zero set, see [9].

Result Two:

Using the moving sphere method we can prove

Theorem 3. *Let K be a non-trivial positive C^1 function in R^2. Assume that K is non-decreasing along each ray $\{t\xi; t \geq 0\}$ for every unit vector $\xi \in S^1$ with $x \cdot \nabla K(x) < 2K(x)$ on R^2 and*

$$K(\infty) = \lim_{|x|\to\infty} K(x) > 0.$$

Then the equation (1) has no smooth solution with $\int_{R^2} Ke^u dx < +\infty$.

We remark that if we have a solution in Theorem 3, then by Cohn-Vossen inequality, we have

$$\int_{R^2} K(x)e^u dx \leq 4\pi. \tag{2}$$

We shall use this fact in our contrary argument.

The plan of the paper is below. We give some asymptotic behavior estimate in section 2. In section 3, we prove the Liouville theorem. In section 4, we prove Theorem 3.

2. Asymptotic behavior

To make the moving plane method get started at infinity, we need to know the behavior of solutions at infinity. Let u be a solution to (1) with the upper bound $u(x) \leq u(0)$.

Let

$$K(x) = K_+(x) - K_-(x),$$

where K_+ be the positive part of K and K_- is the negative part of K.

Define $v = v_0 - v_1$, where

$$v_0(x) = \frac{1}{2\pi}\int_{R^2} \log\frac{|x-y|}{|y|} K_+(y)e^{u(y)}dy$$

and

$$v_1(x) = \frac{1}{2\pi}\int_{R^2} \log\frac{|x-y|}{|y|} K_-(y)e^{u(y)}dy$$

Let $g_0(x) = K_+(x)e^{u(x)}$. Let $R > 0$ be a large number. We write

$$R_+^2 = \{x \in R^2; x_1 \geq 0\} = B_R^+ \bigcup T_1 \bigcup T_2,$$

where $B_R^+(0) = \{x \in R_+^2; |x| \leq R\}$,

$$T_1 = \{y = (y_1, y_2); y_1 > 0, |y - x| \leq \frac{|x|}{2}\},$$

and

$$T_2 = \{y = (y_1, y_2); y_1 > 0, |y - x| \geq \frac{|x|}{2} \text{ and } |y| \geq R$$

Then for $x_1 >> 1$, we have

$$2\pi v_0(x) = (\int_{B_R^+} + \int_{T_1} + \int_{T_2}) \log \frac{|x-y|}{|y|} g_0(y)dy \equiv I_0 + I_1 + I_2.$$

Note that for $y \in T_1$, we have $|y| \le |x|/2$ and $|x-y| \le |y|$. Hence, $I_1 \le 0$. It is clear that

$$I_0 \le \log|x| \int_{B_R^+} g_0(y)dy + C$$

For $y \in T_2$, we have $|y| \ge R > 1$ and $|x-y| \le |x| + |y| \le |x||y|$. Then we have

$$I_2 \le \log|x| \int_{T_2} g_0(y)dy.$$

Hence, we have

$$2\pi v_0(x) \le I_0 + I_2 \le \log|x|(\int_{B_R^+} + \int_{T_2})K_+(y)e^{u(y)}dy + C.$$

It is also clear that for $x_1 << -1$

$$2\pi v_0(x) = (\int_{B_R^+} K_+(y)e^{u(y)}dy + \circ(1)) \log|x|.$$

The lower bound for v_0 can be obtained in the same way (see also the treatment about v_1 below).

We now find the the lower bound for v_1. Define

$$R_-^2 = \{x \in R^2; x_1 \le 0\} = B_R^+ \bigcup S_1 \bigcup S_2,$$

where $B_R^-(0) = \{x \in R_-^2; |x| \le R\}$,

$$S_1 = \{y = (y_1, y_2); y_1 < 0, |y - x| \le \frac{|x|}{2}\},$$

and

$$S_2 = \{y = (y_1, y_2); y_1 < 0, |y - x| \ge \frac{|x|}{2} \ and \ |y| \ge R$$

Then we have that for $x_1 << -1$,

$$2\pi v_1(x) \ge \log|x|(\int_{B_R^-} + \int_{S_2})K_-(y)e^{u(y)}dy + C$$

and $x_1 >> 1$

$$2\pi v_1(x) = (\int_{B_R^-} K_-(y)e^{u(y)}dy + \circ(1)) \log|x| + C.$$

In the same way, we have the same control in x_2 direction.

As for the upper bound of $v_1(x)$ for $|x|$ large, noticing $u(x) \le 0$, we may invoke the Lemma 2.2 and Lemma 2.3 in [6]. For completeness, let us do it here.

Let $g_1(x) = K_-(x)e^{u(x)}$. As before, for $|x| >> 1$, we write

$$2\pi v_1(x) = (\int_{B_R^+} + \int_{S_1} + \int_{S_2}) \log \frac{|x-y|}{|y|} g_1(y)dy \equiv I_0 + I_1 + I_2.$$

For $y \in S_2$, we have $|x-y| \le |x| + |y| \le |x||y|$ and

$$I_2 \le \log \int_{S_2} g_1(y)dy.$$

It is also easy to see that

$$\log|x| \int_{B_R^+} g_1(y)dy - C \le I_0 \le \log|x| \int_{B_R^+} g_1(y)dy + C.$$

Note that $|y| \ge |x|/2 \ge |x-y|$ in S_1, and we have

$$I_1 = (\int_{|x-y|\le|x|^{-4/3}} + \int_{|x|^{-4/3}\le|x-y|\le|x|/2}) \log \frac{|x-y|}{|y|} g_1(y)dy.$$

By computation, we have

$$\int_{|x-y|\le|x|^{-4/3}} |\log \frac{1}{|x-y|}| \le |x|^{-1/2},$$

and

$$|\int_{|x|^{-4/3}\le|x-y|\le|x|/2} \log \frac{|x-y|}{|y|} g_1(y)dy| \le 8\log|x| \int_{|x-y|\le|x|/2} g_1(y)dy.$$

Using $u(x) \le 0$, we have

$$|I_1| \le \epsilon \log|x| + C$$

for $|x|$ large. Hence, we have

$$2\pi v_1(x) \le (\int_{B_R+T_2} g_1(y)dy + \epsilon) \log|x| + C.$$

We remark that for the lower bound of v_1, we use

$$I_0 \ge \log|x| \int_{B_R^+} g_1(y)dy - C$$

and for large $R >> 1$,

$$I_2 \ge -1$$

where we have used the fact $|y| \le 4|x-y|$, which implies that

$$0 \le \log 4 + \log \frac{|x-y|}{|y|}$$

for $y \in S_2$ and

$$I_2 \ge -\log 4 \int_{S_2} g_1(g)dy.$$

Putting all these together, we obtain

Proposition 4. *Assume that*

$$A := \int_{R^2} x_1 e^u > 0.$$

Then there is a positive constant A such that for any $\epsilon > 0$, there exists a large constant $R(\epsilon) > 0$, for $|x| \geq R(\epsilon)$, it holds

$$[A - \epsilon] \log |x| - C \leq 2\pi v(x) \leq [A + \epsilon] \log |x| + C.$$

Recall the following well-known Liouville type theorem for harmonic functions.

Theorem 5. *Assume that w is a harmonic function in R^2 such that*

$$w(x) \leq A \log |x| + C$$

for $|x| \geq R_0$, where R_0 is some fixed positive function. Then the harmonic function $w(x)$ is a constant.

By construction, we have

$$\Delta(u + v) = 0, \quad x \in R^2.$$

Recall that u is bounded from above, so we have

$$u(x) + v(x) \leq A_0 \log |x| + C$$

at infinity for some $A_0 \geq 0$. Then we can use the Liouville Theorem above for harmonic functions to show that $u + v$ is constant.

Theorem 6.

$$u(x) + v(x) = Constant.$$

Remark: Using Theorem 6, we also ave that

$$u(x) \geq -\alpha \log |x| + C.$$

3. MOVING PLANE METHOD

We shall use the moving plane method to prove that $\partial_{x_1} u > 0$. Using the fact that $u(0) = 0$, we then get $u(x_1, 0) = 0$ for all $x_1 \geq 0$, which is a contradiction to the property that $\lim_{x_1 \to \infty} u(x_1, 0) = -\infty$. Then we have proved Theorem 1.

In doing the moving plane method, we let for any real λ,

$$T_\lambda = \{x; x_1 = \lambda\}, \quad \Sigma_\lambda = \{x; x_1 < \lambda\}$$

and

$$x^\lambda = (2\lambda - x_1, x_2).$$

Let

$$w_\lambda = u_\lambda(x) - u(x)$$

where

$$u_\lambda(x) = u(x^\lambda) := u(2\lambda - x_1, x_2).$$

Then we have

$$\Delta w_\lambda(x) = K(x)e^u - K(x^\lambda)e^{u_\lambda}.$$

Claim:

$$w_\lambda(x) > 0, \quad for\ x \in \Sigma_\lambda\ \ and\ \lambda \in \mathbb{R}. \tag{3}$$

We shall prove this Claim in two steps.

Step one. (3) is true for $x \in \Sigma_\lambda$ and $\lambda \leq 0$.

In this step, we first show the following.

Assertion:

$$\Delta w_\lambda(x) < 0,\ whenever\ \ w_\lambda(x) \leq 0\ for\ x \in \Sigma_\lambda, \lambda \leq 0.$$

Suppose that $w_\lambda(x) = u_\lambda(x) - u(x) \leq 0$ for some $x \in \Sigma_\lambda$. If $x_1^\lambda \geq 0$, then $K(x^\lambda) \geq 0$. Hence,

$$\Delta w_\lambda(x) = K(x)e^u - K(x^\lambda)e^{u_\lambda} < 0.$$

If $x_1^\lambda < 0$, then $|x_1^\lambda| \leq |x_1|$ and

$$\Delta w_\lambda(x) = -|K(x)|e^u + |K(x^\lambda)|e^{u_\lambda} \leq 0.$$

Note that near infinity $|x| = +\infty$, by using Theorem 6 and Proposition 4, $w(x)$ is bounded by $2\epsilon \log|x|$.

We now let

$$g_\lambda(x) = \log(\lambda + 3 - x_1) + \log((\lambda + 3 - x_1)^2 + x_2^2), \quad x \in \Sigma_\lambda.$$

Note that

$$\Delta g_\lambda(x) = -(\lambda + 3 - x_1)^{-2} < 0$$

and near $|x| = \infty$, $g_\lambda(x)$ has the behavior no less than $2\log|x|$.

Let

$$\bar{w}(x) = \frac{w_\lambda(x)}{g_\lambda(x)},$$

which is bounded near $|x| = \infty$. Let x_0 be the point such that

$$\bar{w}(x_0) := \inf \bar{w}(x).$$

Such an x_0 can be found since $|\bar{w}(x)|$ is arbitrary small (since ϵ can be small). Consider the function

$$\Delta\bar{w}(x)$$

at x_0. Using

$$\nabla\bar{w}(x_0) = 0, \ \ \Delta\bar{w}(x_0) \geq 0,$$

we obtain that

$$0 \geq \Delta w_\lambda(x_0) = g_\lambda(x_0)\Delta\bar{w}(x_0) + \bar{w}(x_0)\Delta g_\lambda(x_0) > 0,$$

Which is absurd.

Let

$$\lambda_0 = \sup\{\lambda; w_\mu(x) > 0, x \in \Sigma_\mu\ for\ \mu < \lambda\}.$$

Clearly $\lambda_0 > 0$.

Step two. $\lambda_0 = +\infty$.

For otherwise, we assume

$$\lambda_0 < +\infty. \tag{4}$$

By definition, we have a sequence $\lambda_j > \lambda_0$ with $\lim \lambda_j \to \lambda_0$ and

$$\inf_{\Sigma_{\lambda_j}} w_{\lambda_j}(x) < 0.$$

Write

$$w_j(x) = w_{\lambda_j}(x).$$

As in the Step one, we want to show that for large j,

$$\Delta w_j < 0, \ \ whenever \ \ w_j(x) \leq 0, \ for\ x \in \Sigma_{\lambda_j}. \tag{5}$$

Once this is done, we can repeat the Step one to get a contradiction to (4).

We argue by contradiction again. Assume $x_j = (x_{j1}, x_{j2}) \in \Sigma_{\lambda_j}$ such that $w_j(x_j) \leq 0$ and $\Delta w_j(x_j) \geq 0$. for short we write by $\lambda = \lambda_j$. Since $\lambda_j \geq \lambda_0 \geq 0$, $K(x_j^\lambda) > 0$ and

$$0 \leq \Delta w_j(x_j) = K(x_j)e^{u(x_j)} - K(x_j^\lambda)e^{u_{\lambda_j}(x_j)}.$$

Then we have $K(x_j) \geq 0$ and $0 \leq x_{j1} \leq \lambda_j$.

Recall that by continuity, we have

$$w_\lambda(x) \geq 0, \ x \in \Sigma_{\lambda_0}$$

and

$$\partial_{x_1} w_\lambda(x) \geq 0, \ for\ x_1 = \lambda_0.$$

Using the strong maximum principle and Hopf's boundary point lemma, we have

$$w_\lambda(x) > 0, \ x \in \Sigma_{\lambda_0}$$

and

$$\partial_{x_1} w_\lambda(x) > 0, \ for\ x_1 = \lambda_0.$$

If (x_j) is bounded, we may further assume that $x_0 = \lim_{j\to\infty} x_j$. Then either $x_0 \in \Sigma_{\lambda_0}$, which gives $w_{\lambda_0}(x_0) \leq 0$, or $x_0 \in T_{\lambda_0}$ which implies that $|\nabla w_{\lambda_0}(x_0)| = 0$. All these give a contradiction.

Hence (x_j) is unbounded, which implies that x_{j2} is unbounded. Let

$$\phi_j(x) = u_j(x + (0, x_{j2})) - M_j$$

where $M_j = u_j(0, x_{j2}) \to -\infty$. Then we have that ϕ_j is locally bounded from above satisfying

$$-\Delta \phi_j = x_1 e^{M_j} e^{\phi_j}, \ in\ R^2.$$

Using the Harnack inequality, we have a locally uniformly convergent subsequence, still denoted by ϕ_j with its limit ϕ, which satisfies

$$-\Delta \phi = 0, \ in\ R^2.$$

Hence by Liouville theorem, $\phi = 0$.

Since $\phi = 0$, by locally uniformly convergence of (ϕ_j), we have that for any $\epsilon > 0$, there is a j_0 such that for $j > j_0$,

$$\max_{B((x_{j1},0),1+\lambda_0)} |\nabla u_j(x)| < \epsilon.$$

Let

$$H(x) = x_1 e^u - (2\lambda - x_1)e^{u_\lambda}.$$

Then for any $x = (t, x_2)$ where $t \in [x_{j1}, \lambda_j]$,

$$\partial_{x_1} H(x) = e^u(1 + x_1\partial_1 u) + e^{u_\lambda}(1 + x_1^\lambda \partial_1 u^\lambda) > 0.$$

This gives us that for $x_1 < \lambda$,

$$H(x) < H(\lambda, x_2) = 0.$$

Hence,

$$\Delta w_j(x_j) = H(x_j) = (x_1 e^u - (2\lambda - x_1)e^{u_\lambda})|_{x_j} < 0,$$

which yields a contradiction. Thus, (5) is true. By this we have $\lambda_0 = +\infty$ and the Claim is true with $\partial_{x_1} u > 0$. In particular, $\partial_{x_1} u(0,0) > 0$.

Since $u(0,0) = 0$ is the maximum of u, we have

$$\partial_{x_1} u(0,0) = 0.$$

A contradiction. Hence no such solution u exists.

4. MOVING SPHERE METHOD

In this section, we prove Theorem 3.

Proof. Without loss of generality, we may assume that $K(\infty) = 1$. We shall use the method of moving spheres, which is a little bit easier than the moving plane method since we only use the maximum principle in bounded balls.

Let

$$v(x) = u(\frac{x}{|x|^2}) - 4\log|x|.$$

For $\lambda > 0$,

$$v_\lambda(x) = u(\lambda^2 \frac{x}{|x|^2}) + 4\log\lambda - 4\log|x|.$$

Then $v_\lambda(x)$ satisfies

$$\Delta v_\lambda + K(\lambda x/|x|^2)exp(v_\lambda) = 0.$$

Let $w_\lambda(x) = v_\lambda(x) - u(x)$. Then w_λ satisfies

$$\Delta w_\lambda(x) + b_\lambda(x) w_\lambda(x) = Q_\lambda(x)$$

where

$$b_\lambda(x) w_\lambda(x) = K(\lambda x/|x|^2)(\frac{exp(v_\lambda) - exp(u)}{v_\lambda - u})$$

and

$$Q_\lambda(x) = (K(\lambda x) - K(\lambda x/|x|^2))exp(u)$$

By our monotone assumption on K, we have $Q_\lambda(x) \le 0$ for $|x| < 1$ and $\lambda > 0$.

We claim that

$$w_\lambda(x) > 0,\ |x| < 1 \ \ and\ \lambda > 0. \tag{6}$$

Assume that $w_\lambda(x_0) = \inf_{B_1} w_\lambda < 0$ for some $x_0 \in B_1$ and some small $\lambda \le 1$, we have

$$\Delta w_\lambda(x_0) \le 0.$$

But this is impossible by strong maximum principle. Hence (6) is true for small $\lambda > 0$. Let

$$\lambda_1 = \sup\{\lambda; w_\mu(x) > 0 \ for\ all\ 0 < \mu \le \lambda \ and\ |x| < 1\}.$$

Then we have

$$\lambda_1 = +\infty.$$

For otherwise, we have $\lambda_1 < \infty$. Note that w_{λ_1} may have singularity in $x = 0$. Using the maximum principle to $w_{\lambda_1} \ge 0$ in $0 < |x < 1$, we have $w_{\lambda_1}(x) > 0$. Hence for any small $\epsilon > 0$, there is exists a $\delta > 0$ such that for $|\lambda - \lambda_1| < \delta$, we have

$$\inf_{|x|=\epsilon} w_\lambda(x) > 0.$$

Arguing as above (as for small λ before the definition of λ_1), we have that for $|\lambda - \lambda_1| < \delta$ and $0 < |x| \le \epsilon$,

$$w_\lambda(x) > 0, \ for\ 0 < |x| \le \epsilon.$$

By the definition of λ_1, we have $\lambda_n > \lambda_1$ and $0 \ne x_n \in B_1$ such that $\lambda_n \to \lambda_1$ and

$$w_{\lambda_n}(x_n) = \inf_{B_1} w_{\lambda_n} < 0.$$

With loss of generality, we can let $x_0 = \lim_n x_n$. Then $|x_0| = 1$ and $\nabla w_{\lambda_1}(x_0) = 0$, which is a contradiction to Hopf's boundary point lemma. Hence the Claim (6) is true. The Claim (6) and the Hopf boundary lemma imply that, for $y = x/|x|^2$,

$$v(x/|x|^2) - 2\log|x| = v(y) + 2\log|y|$$

is monotone decreasing along each ray, i.e., for $t > 0$,

$$\partial_t(v(ty) + 2\log t|y|) \le 0. \tag{7}$$

We now claim that there exists a uniform constant $C > 0$ such that

$$u(x) + 4\log|x| \le C \tag{8}$$

in R^2. For otherwise, we have a sequence $(x_j) \subset R^2$ such that

$$|x_j| \to \infty$$

and

$$u(x_j) + 4\log|x_j| \to +\infty.$$

That is saying that for $y_j = x_j/|x_j|^2$, $|y_j| \to 0$,

$$v(y_j) \to +\infty.$$

For $R > 0$, define

$$S_j(y) = (R^2 - |y - y_j|^2)e^{v(y)},$$

in $B(y_j, R)$. Note that

$$S_j(\bar{y}_j) = \max_{B(y_j,R)} S_j(y) \geq S_j(y_j) = exp(v(y_j) + 2\log R) \to +\infty.$$

Define $M_j = -2\log\delta_j = v(\bar{y}_j)$ and

$$\bar{v}_j(\eta) = v(\bar{y}_j + \delta_j y) - M_j.$$

Using the standard blowing up method (see [10]) we know that the renormalization $\bar{v}_j$ uniformly converges to the bubble solution U_0 in $C^2_{loc}(R^2)$. Using the symmetry of U_0 at η_0, we know that $\bar{v}_j$ has a local maximum at η_j with $\lim_j \eta_j = \eta_0$. Here η_j is the point such that $\bar{y}_j = y_j + \delta_j\eta_j$. Going back to v, v has a local maximum at $y{*}_j = y_j + \delta_j\eta_j$. Then we have at $t = 1$,

$$\partial_t(v(ty{*}_j) + 2\log t) \leq 0, \ and \ \partial_t v(ty{*}_j) = 0,$$

which are contrary each other.

Since $K(\infty) = 1$, we may show that

$$u(x) \geq -4\log|x| + c \tag{9}$$

for large $|x| >> 1$. In fact, there is another way. According to Lemma 2.2 in [6], we have

$$u(x) \geq -\beta\log|x| + C_1$$

for $\beta = \frac{1}{\pi}\int Ke^u dx$ and $|x|$ large. By Cohn-Vossen inequality (2) we have $\beta \leq 4$. Then we have

$$-4\log|x| + c \geq u(x) \geq -\beta\log|x| + C_1,$$

which implies that

$$(4-\beta)\log|x| \leq C, \quad for \quad |x| >> 1$$

and then $\beta \geq 4$. Hence, $\beta = 4$. Combining (8) with (9), we get

$$u(x) = -4\log|x| + q(x)$$

with $q(x)$ being bounded at $|x| \to \infty$. Applying the Pohozaev identity in the appendix (see also [11])

$$-\int_{\partial B_R}[R(|\nabla u|^2/2 - |\partial_r u|^2 - Ke^u)] = \int_{B_R}(K + \frac{1}{2}x\cdot\nabla K)e^u$$

to u in a large ball B_R and arguing as in pages 136-137 in [6] we then get

$$\lim_{R\to\infty}\int_{B_R} x\cdot\nabla K e^u = 16\pi.$$

By our assumption $x\cdot\nabla K < 2K$, we get a contradiction. Then we are done. $\square$

5. APPENDIX

In this appendix, we present the Pohozaev identity for equation (1). This fact is well-known to experts (see [11]). Since the proof is different from the higher dimensional case, we give a proof below.

Let $F(x,u) = K(x)e^u$. Integrating both sides of (1) over $B_R(0)$, we have

$$-\int_{\partial B_R(0)} \mu_j u_j = \int K e^u.$$

Here $\int = \int_{B_R(0)}$ and $\mu_j = x_j/R$.

Multiplying both sides of (1) by $r\partial_r u = x_j \partial_j u$ and integrating over $B_R(0)$, we have

$$\int (x_j\partial_j u)_i u_i - \int_{\partial B_R(0)} x_j u_j \nu_i u_i = \int K x_j \partial_j e^u.$$

Note that

$$\int (x_j\partial_j u)_i u_i = \int (|\nabla u|^2 + x_j u_{ij} u_i)$$

and

$$\int x_j u_{ij} u_i = \frac{1}{2}\int x_j (|\nabla u|^2)_j = -\int |\nabla u|^2 + \frac{1}{2}\int_{\partial B_R(0)} x_j \nu_j |\nabla u|^2.$$

Then we have

$$\int (x_j\partial_j u)_i u_i = \frac{1}{2}\int_{\partial B_R(0)} x_j \nu_j |\nabla u|^2.$$

Since

$$\int K x_j \partial_j e^u = -\int (2Ke^u + x_j \partial_j K e^u) + \int_{\partial B_R(0)} x_j \nu_j K e^u,$$

we have

$$\frac{1}{2}\int_{\partial B_R(0)} x_j \nu_j |\nabla u|^2 - \int_{\partial B_R(0)} x_j u_j \nu_i u_i - \int_{\partial B_R(0)} x_j \nu_j K e^u = -\int (2Ke^u + x_j \partial_j K e^u).$$

This is the standard *Pohozaev identity* for the mean field equation in $B_R(0)$. Sometimes, people would like to use another form of it, which is

$$\frac{1}{2}\int_{\partial B_R(0)} x_j \nu_j |\nabla u|^2 - \int_{\partial B_R(0)} x_j u_j \nu_i u_i - \int_{\partial B_R(0)} x_j \nu_j K e^u$$
$$-2\int_{\partial B_R(0)} \mu_j u_j = -\int x_j \partial_j K e^u.$$

Similarly, by multiplying by u_j, we can get another Pohozaev identity:

$$\int \partial_j K e^u = \int_{\partial B_R(0)} (\nu_k u_k u_j - \frac{1}{2}|\nabla u|^2 \nu_j + K e^u \nu_j) \tag{10}$$

All these Pohozaev identities are useful in the study of mean field equation.

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